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APPROXIMATE SOLUTION OF NONLINEAR FRACTIONAL INTEGRO-DIFFERENTIAL EQUATIONS BY HE'S HOMOTOPY PERTURBATION METHOD AND THE MODIFICATION OF HE'SVARIATIONAL ITERATION METHOD

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ABSTRACT

In this paper, we compare the modification of He's variational iteration method (MVIM), and He's homotopy perturbation method (HPM), in order to obtain the approximate solution of nonlinear fractional integro-differential equations of Volterraand Fredholmintegro-differential equations, we presentsome examples to find out accuracy of the methods.

KEYWORDS: Fractional Integro-Differential Equations, Caputo Derivative, Modification of He'svariationa literation Method, Homotopy Perturbation Method

INTRODUCTION

The fractional integro-differential equations is a special kind of equations collecting integro-differential equations and calculus,([17],[18]). In recent years, there has been a growing mathematical formulations of physical phenomena, such as nonlinear fractional analysis and their applications in the theory of Engineering, Mechanics, Physics, Chemical kinetics, Astronomy, Biology Economics, potential theory and Electrostatics contain integro-differential equations, ([3],[6],[15],[16]).

The variational iteration method was first proposed by He, ([1],[2],[13]), and was been worked out over a number of years by many authors. This method has been shown to effectively, easily and accurately solve a large class of nonlinear problems.In this paper our propose the reliable modification of He's VIM (MVIM), that was introduced by Gharbani, [1], for solving the nonlinear fractional integro-differential equations by constructing an initial trial-function without unknown parameters so that one iteration leads to exact solution. The other propose of this paper we study He's perturbation method, [7], for approximating the solution of nonlinear fractional integro-differential equations.Wewell consider fractional order integro-differential equations of the form:

$$\mathbf{D}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) + \boldsymbol{\lambda} \int_0^x \mathbf{k}(\mathbf{x},\mathbf{t})\mathbf{F}\big(\mathbf{y}(\mathbf{t})\big)\mathbf{dt} \tag{1.1}$$

and

$$\mathbf{D}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) + \boldsymbol{\lambda} \int_0^x \mathbf{k}(\mathbf{x},\mathbf{t})\mathbf{F}\big(\mathbf{y}(\mathbf{t})\big)\mathbf{dt} \tag{1.2}$$

for x; t $\in$[0; 1], $\lambda$is a numerical parameter, where the function g(x), k(x; t) are known and y(x) is the unknown function, $D^{\alpha}$is Caputo's fractional derivative and $\alpha$is a parameter describing the order of thefractional derivative and $F\big(y(t)\big) = f(y(t))^q$, q >1,is a nonlinear continuous function.

BASIC DEFINITIONS

In this section we present some basic definitions and properties of the fractional calculus theory, which are used in this paper, ([5],[9],[14]).

Definition: 2.1

A real function y(x),x >0, is said to be in the space C_μ, μ∈R, if there exists a real numberp >μ, such thaty(x) = $x^p y_1(x)$, where y1(x) ∈C[0; 1). Clearly $C_\mu < C_\beta$ if β<μ

Definition: 2.2

A function y(x), x >0, is said to be in the spaceC_μ^m , mm ∈ N ∪ {0}, if $y^{(m)} \in C_\mu$

Definition: 2.3

The left sided Riemann-Liouville fractional integral operatorα ≥ 0, of a function, ∈ C_μ , μ≥-1, is defined as:

$$\mathbf{I^\alpha y(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1} y(t)dt}, \qquad \boldsymbol{\alpha>0\ , x>0} \tag{2.1}$$

$$\mathbf{I^\alpha y(x) = y(x)}, \qquad (\mathbf{I^0 = I\ idendity\ operator}) \tag{2.2}$$

Definition: 2.4

Let y ∈ C_{-1}^m, m ∈ N ∪ {0}, then the Caputo's fractional derivative of y(x) is definition as:

$$\mathbf{y(x)} = \begin{cases} \mathbf{J^{m-\alpha} y^m(x)}, & \mathbf{m-1 < m, m \in N} \\ \mathbf{\frac{D^m y(x)}{Dx^m}}, & \boldsymbol{\alpha = m} \end{cases} \tag{2.3}$$

Hence, we have the following properties:

$\mathbf{I^\alpha y(x) I^\alpha y(x) = I^{\alpha+\beta} y(x)}$, **for all** $\boldsymbol{\alpha, \beta \geq 0}, \mathbf{y \in C_\mu}, \boldsymbol{\mu} > 0$

$\mathbf{I^\alpha x^\gamma \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+1-\alpha)} x^{\gamma-\alpha}}$,

for $\mathbf{x} > 0$, $\boldsymbol{\alpha \geq 0}$, $\boldsymbol{\gamma > -1}$

$\mathbf{I^\alpha D^\alpha y(x) = y(x) - \sum_{i=0}^{m-1} y^{(k)}(0^+) \frac{x^k}{k!}}$, $\mathbf{x} > 0.$

And Caputo fractional differentiation is a linear operation, similar to inter order differentiation.

$\mathbf{D^\alpha[\lambda\, y(x) + \mu\, g(x)] = \lambda D^\alpha y(x) + \mu\, D^\alpha g(x)}$,whereλ and μ are constants.

Numerical solution of nonlinear Volterra and Fredholm fractional integro-differential equations

In this section the Modified of He's Variational iteration Method and He's Homotopy Perturbationmethod are applied for solving nonlinear fractional integro-differential equations**.**

The Modified of He's Variational iteration Method

In the first we will propose the reliable modification of the (VIM),[1], for solving nonlinear fractional

Integro- differential equations withe initial conditions by constructing an initial trial-function without unknown parameters. Here, we consider the following fractional functional equation,

$$\mathbf{y(x) = Ry(x) + Ny(x) = g(x)}, \tag{3.1}$$

where$L_x = D^x$, R is a linear differential operator, N represents the nonlinear terms, and g is the sourceterm. By using (2.6) applying the inverse L_x^{-1} to both sides of (3.1) we obtain:

$$\mathbf{y(x) = f(x) - L_x^{-1}[Ry(x) - L_x^{-1}[Ny(x)]}, \tag{3.2}$$

where $L_x^{-1} = I^{\alpha}$, and $L_x^{-1}[g(x)] = f(x)$.Can be applied in the above equations (1.1) and (1.2) and using the basic character ofHe' s method is construction of a correction fractional for (3.1) which reads,

$$\mathbf{y_{n+1}(x) = y_n(x) + \int_0^x \lambda(t)[Ly_N(t) + R\breve{y}_N(t) + N\breve{y}_N(t) - g(t)]dt}, \tag{3.3}$$

and

$$\mathbf{y_{n+1}(x) = y_n(x) + \int_0^1 \lambda(t)[Ly_N(t) + R\breve{y}_N(t) + N\breve{y}_N(t) - g(t)]dt}, \tag{3.4}$$

whereλCan be applied in the above equations (1.1) and (1.2) and using the basic character of He s method is construction of a correction fractional for (3.1) which reads,approximate solution of (1.1) or (1.2), and $\breve{y}_n$denotes a restricted Variation, $\delta\breve{y}_n = 0$, to solve equations (3.3) and (3.4) use a Lagrange multiplier resulting from the integration by parts. Then the successive approximations $y_n(x), n \geq 0$of the solution y(x) can obtain by using a Lagrange multiplier and by using any selective function$y_0(x)$, ([10],[11],[12]). The exact solution may be obtain by using,

$$\mathbf{\lim_{x\to\infty} y_n(x) = y(x)}. \tag{3.5}$$

As a result, we have the following Variational iteration formula for (3.2),

$$\begin{cases} \mathbf{y_0(x)\text{is an arbitrary initial guess}}, \\ \mathbf{y_{n+1}(x) = f(x) - L_x^{-1}[Ry_n(x)] - L_x^{-1}[Ny_n(x)]} \end{cases} \tag{3.6}$$

The MVIM, that was introduced by Ghorbai et al, [1], can be established based on the assumption that thefunction f(x) of the iterative relation (3.6) can be divided into two parts, namely$f_0(x)$ and $f_1(x)$, then we set,

$$\mathbf{f(x) = f_0(x) + f_1(x)} \tag{3.7}$$

According to the assumption (3.7) and by the relationship (3.6), we construct the following Variational iteration formula,

$$\begin{cases} \mathbf{y_0(x) = f_0(x)}, \\ \mathbf{y_1(x) = f(x) - L_x^{-1}[Rf_0(x)] - L_x^{-1}[Nf_0(x)]}, \\ \mathbf{y_{n+1}(x) = f(x) - L_x^{-1}[Ry_n(x)] - L_x^{-1}[y_n(x)]}. \end{cases} \tag{3.8}$$

He's Homotopy perturbation method

The basic consider of homotopy perturbation method illustrated by consider the following nonlinear

functional equation,([4],[8]).

$$\mathbf{A(u) = y(x)}, \tag{3.9}$$

with the following boundary conditions, $\left(\mathrm{U}, \frac{\partial u}{\partial n}\right) = 0, x \in \Gamma$, where A is a general functional operator, U is a boundary operator, y(x) is a known analytic function, and Γ is the boundary of domain Ω , the operator A can be decomposed into two parts L and N, where L is linear and N is a nonlinear operator, equation (3.9) can by rewritten as the following:

$$\mathbf{L(U) + N(U) - y(x) = 0} \tag{3.10}$$

We construct a homotopy$V(x,p): \Omega \times [0,1] \to R$, which satisfies:

$$\mathbf{H(V,p) = (1-p)[L(V) - L(U_0)] + p[A(V) - y(x)] = 0} \tag{3.11}$$

Where $p \in [0,1], x \in \Omega$,

or

$$\mathbf{H(V,p) = L(V) - L(U_0) + pL(U_0) + p[N(V) - y(x)] =} \tag{3.12}$$

where$y_0$is an initial approximation for the solution of equation (3.9). In this method, we use the homotopy parameter p to expand:

$$\mathbf{V = V_0 + pV_1 + p^2V_2 + \cdots} \quad . \tag{3.13}$$

The approximate will be obtained by taking the limit as p tends to 1,

$$\mathbf{U = \lim_{p \to 1} V = V_0 + V_1 + V_2 + \cdots} \tag{3.14}$$

To illustrate for equations (1.1) or (1.2) substituting (3.11) into (1.1) or (1.2) we obtain:

$$\mathbf{D^\alpha y_i(x) = p(g_i(x) + \lambda \int_0^x k(x,t)F(y(t)dt)}, \tag{3.15}$$

or

$$\mathbf{D^\alpha y_i(x) = p(g_i(x) + \lambda \int_0^1 k(x,t)F(y(t)dt)}, \tag{3.16}$$

We expand the solution of equations (1.1) or (1.2) in the following form:

$$\mathbf{y_i(x) = \sum_{i=0}^{\infty} p^i y_i(x) = y_0(x) + py_1(x) + p^2y_2(x) + \cdots} \quad . \tag{3.17}$$

Substituting (3.17) into (3.15) or (3.16) and collecting the terms with the same powers ofp, we obtain aseries of equations of the form:

$$\mathbf{p^0: D^\alpha y_0(x) = 0},$$

$$\mathbf{p^1: D^\alpha y_1(x) = g(x) + \lambda \int_0^x k(x,t)F(y(t))dt},$$

$$\mathbf{p^2: D^\alpha y_2(x) = g(x) + \lambda \int_0^x k(x,t)F\big(y(t)\big)dt},$$

or

$$\mathbf{p^0: D^\alpha y_0(x) = 0},$$

$$\mathbf{p^1: D^\alpha y_1(x) = g(x) + \lambda \int_0^1 k(x,t) F(y(t)) dt},$$

$$\mathbf{p^2: D^\alpha y_2(x) = g(x) + \lambda \int_0^1 k(x,t) F(y(t)) dt},$$

that these equations can be easily solved by applying the operator $I^\alpha$the inverse of the operator D^α according to equation (2.6), that is by setting p = 1, in equations (3.15) or (3.16) we can entirely determine setting p = 1, in equations (3.15) or (3.16) we can entirely determine the (HPM) series solutions,([4],[8]).

$$\mathbf{y(x) = \sum_{i=0}^{\infty} y_i(x)}. \tag{3.18}$$

NUMEICAL EXAMPLES

In this section we present some numerical examples of nonlinear fractional integro-differential equations by the modification of He's Variational iteration method and He's Homotopy perturbation method.

Example 4.1

Consider the following nonlinear fractional integro-differential equation:

$$\mathbf{D^{0.9} y(x) = g(x) + \int_0^x (x-t)^2 [y(t)]^3\, dt}, \tag{4.1}$$

Where $g(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{\frac{2}{5}}\Gamma(\frac{2}{5})} - \frac{16}{315}x^{\frac{2}{5}}$, with the initial condition y(0) = 0, and exact solution y(0)=$\sqrt{x}$

The solution according to (MVIM)

$$\mathbf{D^{0.9} y(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{\frac{2}{5}}\Gamma(\frac{2}{5})} - \frac{16}{315}x^{\frac{2}{5}} + \int_0^x (x-t)^2 [y(t)]^3\, dt}, \tag{4.1}$$

We take the operator$I^{9/10}$, on both sides of equation (4.1) we obtain:

$$\mathbf{y(x) = y(0) - \sum_{k=0}^{m-1} y^{(k)}(0^+)\frac{x^k}{k!} + I^{9/10}\left(\frac{1}{2}\frac{\sqrt{\pi}}{x^{\frac{2}{5}}\Gamma(\frac{2}{5})} - \frac{16}{315}x^{\frac{2}{5}} + \int_0^x (x-t)^2 [y(t)]^3\, dt\right)} \tag{4.2}$$

According to the original VIM (3.3) and corresponding the recursive scheme (3.6), we obtain:

$$\mathbf{f(x) = f_0(x) + f_1(x) = I^{9/10}\left(\frac{1}{2}\frac{\sqrt{\pi}}{x^{\frac{2}{5}}\Gamma(\frac{2}{5})} - \frac{16}{315}x^{\frac{2}{5}}\right)}, \tag{4.3}$$

$$\mathbf{f(x) = \sqrt{x} - 0.01103948449x^{27/5}}, \tag{4.4}$$

by assuming

$$\mathbf{f(x) = \sqrt{x},\ f_1(x) = -0.01103948449x^{27/5}}, \tag{4.5}$$

with starting of the initial approximation,$y(x) = f_0(x) = \sqrt{x}$, we obtain:

$$\mathbf{y_0(x) = \sqrt{x}} \tag{4.6}$$

$$\mathbf{y_1(x) = \sqrt{x} - 0.01103948449x^{27/5} + L_x^{-1}}, \tag{4.7}$$

$$\mathbf{y_1(x) = \sqrt{x} - 0.01103948449x^{27/5} + I^{9/10}\left(\int_0^x (x-t)^2 [y(t)]^3\, dt\right)} \tag{4.8}$$

$$\mathbf{y_1(x) = \sqrt{x}}.$$

$$\mathbf{y_{n+1}(x) = \sqrt{x} - 0.01103948449x^{27/5} + L_x^{-1}[y_n(x)] = \sqrt{x},\ n \geq 1} \tag{4.9}$$

in similarly view equation (4.7) it is obtained$y(x) = \sqrt{x}$, where it is the exact solution of equation (4.1).

The solution according to (HPM)

$$\mathbf{D^{0.9}y(x) = \frac{1}{2}\frac{\sqrt{\pi}}{x^{\frac{2}{5}}\Gamma(\frac{2}{5})} - \frac{16}{315}x^{\frac{2}{5}} + \int_0^x (x-t)^2 [y(t)]^3\, dt}, \tag{4.1}$$

According to (3.16) we construct the following homotopy:

$$\mathbf{D^{9/10}y(x) = p\left(g(x) + \int_0^x (x-t)^2 [y(t)]^3\, dt\right)}, \tag{4.10}$$

substituting (3.13) into (4.9) we obtain:

$$\mathbf{p^0: D^{9/10}y_0(x) = 0},$$

$$\mathbf{p^1: D^{9/10}\, y_1(x) = g(x) + \int_0^x (x-t)^2\, [y_0(t)]^3 dt},$$

$$\mathbf{p^2: D^{9/10}y_2(x) = \int_0^x (x-t)^2 [3(y_0(t))^2 y_1(t)] dt},$$

$$\mathbf{p^3: D^{9/10}y_3(x) = \int_0^x (x-t)^2 [3(y_0(t))^2 y_2(t) + 3y_0(t)(y_1(t))^2] dt},$$

$$\mathbf{p^4: D^{9/10}y_4(x) = \int_0^x (x-t)^2\, [3(y_0(t))^2\, y_3(t) + 6y_0(t) + y_1(t) + (y_3(t))^2] dt},$$

by applying the operators$I^{9/10}$to the above sets we obtain:

$$\mathbf{y_0(x) = 0}$$

$$\mathbf{y_1(x) = \sqrt{x} - 0.01103948449x^{27/5}}$$

$$\mathbf{y_2(x) = 0},\quad \mathbf{y_3(x) = 0}, \ldots$$

$$\mathbf{y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) + y_1(x) + y_2(x) + \cdots} \quad .$$

Therefore the approximate solution of (4.1),

$$\mathbf{y(x) \cong \sqrt{x} - 0.01103948449x^{27/5}}.$$

Table 1, Figure 1 and 2 shown the numerical result of example 4.1 .

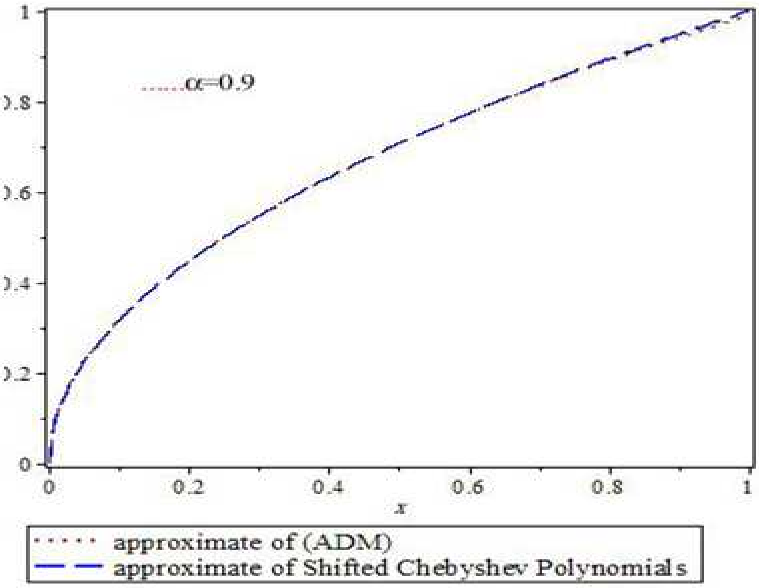

Figure 1: Numerical Result of Example 4.1

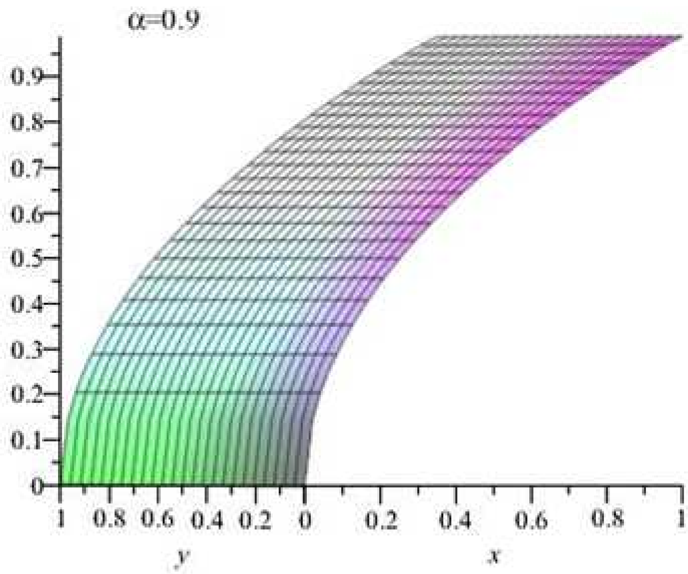

Figure 2: Approximate Solution of Example 4.1

Table 1: Indicate The Amount of Error in Example 4:1

x	exact = (MV IM)	Approximant by(HPM)	Error of(HPM)
0.1	0.3162277660	0:3162277221	4.39×10^{-8}
0.2	0.4472135955	0:4472117398	0:0000018557
0.3	0.5477225575	0:5477059844	0:0000165731
0.4	0.6324555320	0:6323771759	0:0000783561
0.5	0.7071067812	0:7068453323	0:0002614489
0.6	0.7745966692	0:7738968827	0:0006997865
0.7	0.8366600265	0:8350513148	0:0016087117
0.8	0.8944271910	0:8911186637	0:0033085273
0.9	0.9486832981	0:9424336100	0:0062496881
1	1.0	0:9889605155	0:0110394845

Example 4.2

Consider the following nonlinear fractional integro-differential equation:

$$\mathbf{D}^{1/3}\,\mathbf{y}(\mathbf{x}) = \frac{9}{5}\frac{\mathbf{x}^{5/3}}{\Gamma(\frac{2}{3})} - \frac{7}{3}\,\mathbf{x} + \frac{1}{4}\int_0^1 \mathbf{x}(\mathbf{1}-\mathbf{t})[\mathbf{y}(\mathbf{t})]^2\mathbf{dt}, \tag{4.10}$$

we take the operator $I^{1/3}$on both sides of equation (4.10) we obtain:

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}(\mathbf{0}) - \sum_{\mathbf{i=0}}^{\mathbf{m-1}} \mathbf{y}^{(\mathbf{k})}\,(\mathbf{0}^+)\frac{\mathbf{x}^\mathbf{k}}{\mathbf{k!}} + \mathbf{I}^{1/3}(\frac{9}{5}\frac{\mathbf{x}^{5/3}}{\Gamma(\frac{2}{3})} - \frac{7}{3}\,\mathbf{x} + \frac{1}{4}\int_0^1 \mathbf{x}(\mathbf{1}-\mathbf{t})[\mathbf{y}(\mathbf{t})]^2\mathbf{dt}), \tag{4.11}$$

According to the original VIM (3.3) and corresponding the recursive scheme (3.6) we obtain:

$$\mathbf{f}(\mathbf{x}) = \mathbf{f_0}(\mathbf{x}) + \mathbf{f_1}(\mathbf{x}) = \mathbf{1} + \mathbf{I}^{1/3}\left(\frac{9}{5}\frac{\mathbf{x}^{5/3}}{\Gamma(\frac{2}{3})} - \frac{7}{3}\,\mathbf{x}\right), \tag{4.12}$$

$$\mathbf{f}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2 - \mathbf{0.1469798559}\mathbf{x}^{4/3}, \tag{4.13}$$

by assuming,

$$\mathbf{f_0}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2, \qquad \mathbf{f_1}(\mathbf{x}) = -\mathbf{0.1469798559}\mathbf{x}^{4/3},$$

with starting the initial approximation, $y_0(x) = f_0(x) = 1 + x^2$,we obtain:

$$\mathbf{y_0}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2,$$

$$\mathbf{y_1}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2 - \mathbf{0.1469798559}\mathbf{x}^{4/3} + \mathbf{L_x^{-1}}[\mathbf{f_0}(\mathbf{x})],$$

$$\mathbf{y_1}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2 - \mathbf{0.1469798559}\mathbf{x}^{4/3} + \mathbf{I}^{1/3}(\int_0^1 \mathbf{x}(\mathbf{1}-\mathbf{t})[\mathbf{y}(\mathbf{t})]^2\mathbf{dt},$$

$$\mathbf{y_1}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2$$

$$\mathbf{y_{n+1}}(\mathbf{x}) = \mathbf{1} + \mathbf{x}^2 - \mathbf{0.1469798559}\mathbf{x}^{4/3} + \mathbf{L_x^{-1}}[\mathbf{y_n}(\mathbf{x})] = \mathbf{1} + \mathbf{x}^2, \mathbf{n} \geq \mathbf{1} \tag{4.14}$$

in similarly view equation (4.14) it is obtained, $y(x) = 1 + x^2$, where it is the exact solution of (4.10).

The solution according to (HPM)

$$\mathbf{D}^{1/3}\,\mathbf{y}(\mathbf{x}) = \frac{9}{5}\frac{\mathbf{x}^{5/3}}{\Gamma(\frac{2}{3})} - \frac{7}{3}\,\mathbf{x} + \frac{1}{4}\int_0^1 \mathbf{x}(\mathbf{1}-\mathbf{t})[\mathbf{y}(\mathbf{t})]^2\mathbf{dt}, \qquad (4.10)$$

According to (3.16) we construct the following homotopy:

$$\mathbf{D}^{1/3}\,\mathbf{y}(\mathbf{x}) = \mathbf{p}(\mathbf{g}(\mathbf{x}) + \frac{1}{4}\int_0^1 \mathbf{x}(\mathbf{1}-\mathbf{t})[\mathbf{y}(\mathbf{t})]^2\mathbf{dt}). \tag{4.15}$$

Substituting (3.13) into (4.14),

$$\mathbf{p}^0:\ \mathbf{D}^{1/3}\mathbf{y_0}(\mathbf{x}) = \mathbf{0}$$

$$\mathbf{p}^1: \mathbf{D}^{1/3}\mathbf{y_1}(\mathbf{x}) = \mathbf{g}(\mathbf{x}) + \frac{1}{4}\int_0^1 \mathbf{x}(\mathbf{1}-\mathbf{t})[\mathbf{y_0}(\mathbf{t})]^2\mathbf{dt}$$

$$\mathbf{p^2: D^{1/3}\, y_0(x) = \frac{1}{4}\int_0^1 x(1-t)[2y_0(t)y_1(t)]dt},$$

$$\mathbf{p^3: D^{1/3} y_3(x) = \frac{1}{4}\int_0^1 x(1-t)[2y_0(t)y_2(t) + (y_1(t))^2]dt},$$

$$\mathbf{p^4: D^{1/3} y_4(x) = \frac{1}{4}\int_0^1 x(1-t)[2y_0(t)y_3(t) + 2y_2(t)y_1(t)]},$$

by applying the operators $I^{1/3}$to the above sets we obtain:

$$\mathbf{y_0(x) = 1},$$

$$\mathbf{y_1(x) = x^2 - 0.04199424454x^{4/3}},$$

$$\mathbf{y_2(x) = 0.03272782533x^{4/3}},$$

$$\mathbf{y_3(x) = 0.008024681284x^{4/3}},$$

$$\mathbf{y_4(x) = 0.0009942270254\, x^{4/3}},$$

There for the approximations to the solution of equation (4.10) will be determined as

$$\mathbf{y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) + \cdots} \quad .$$

$$\mathbf{y(x) \cong 1 + x^2 - 0: 00024751090x^{4/3}},$$

Table 2, Figure 3 and 4 shown the numerical result of example 4.2 .

Table 2: Indicate the amount of error in Example 4.2

x	exact = (MV IM)	Approximant by(HPM)	Error of(HPM)
0.1	1.01	1.009988512	0.000011488
0.2	1.04	1.039971051	0.000028949
0.3	1.19	1.089950292	0.000049708
0.4	1.16	1.159927053	0.000072947
0.5	1.25	1.249901775	0.000098225
0.6	1.36	1.359874745	0.000125255
0.7	1.49	1.489846164	0.000153836
0.8	1.64	1.639816185	0.000183815
0.9	1.81	1.809784928	0.000215072
1	2.00	1.999752489	0.000247511

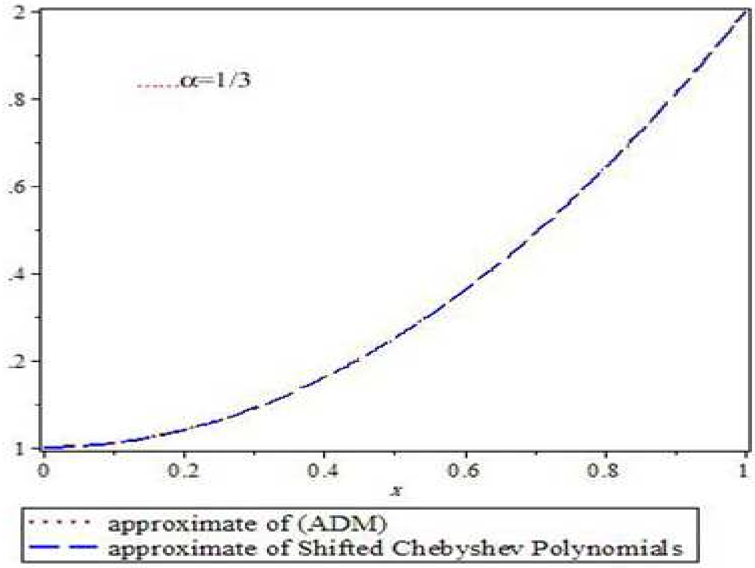

Figure 3: Numerical Result of Example 4.2

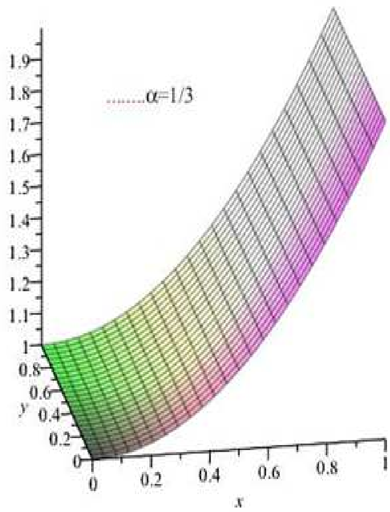

Figure 4: Approximate solution of example 4.2

Example 4.3

Consider the following nonlinear fractional integro-differential equation:

$$\mathbf{D}^{2/3}\,\mathbf{y}(\mathbf{x}) = \frac{\mathbf{81}\,\mathbf{x}^{7/3}\sqrt{3}\,\Gamma(2/3)}{\mathbf{28}\,\boldsymbol{\pi}} - \frac{\sqrt{\mathbf{x}}}{\mathbf{8}} + \int_0^1 \sqrt{x}\,t\,[y(t)]^2\,dt\,, \tag{4.15}$$

with the initial condition$y(0) = 0$and exact solution $y(x) = x^3$.

The solution according to(MVIM)

Now we take the operator $I^{2/3}$on both sides of equation (4.14) we obtain:

$$\mathbf{y}(\mathbf{x}) = \mathbf{y}(\mathbf{0}) - \sum_{k=0}^{m-1} \mathbf{y}^{(k)}(\mathbf{0}^+)\,\frac{\mathbf{x}^k}{\mathbf{k}!} + \mathbf{I}^{2/3}\left(\frac{\mathbf{81}\,\mathbf{x}^{7/3}\sqrt{3}\,\Gamma(2/3)}{\mathbf{28}\,\boldsymbol{\pi}} - \frac{\sqrt{\mathbf{x}}}{\mathbf{8}} + \int_0^1 \sqrt{\mathbf{x}}[\mathbf{y}(\mathbf{t})]^2\mathbf{dt}\right), \tag{4.16}$$

according to the original VIM (3.3) and corresponding the recursive scheme (3.6) we obtain:

$$\mathbf{f(x) = f_0(x) + f_0(x) = I^{2/3}\left(\frac{81\, x^{7/3}\sqrt{3}\,\Gamma(2/3)}{28\qquad \pi} - \frac{\sqrt{x}}{8}\right)}, \qquad (4.17)\text{f(x)}$$

$$= \mathbf{x^3 - 0.1023508743\, x^{7/6}},$$

by assuming

$$\mathbf{f_0(x) = x^3}, \qquad \mathbf{f_1(x) = -0.1023508743\, x^{7/6}}, \qquad (4.18)$$

with starting of the initial approximation , $y_0(x) = f_0(x) = x^3$,we obtain:

$$\mathbf{y_0(x) = x^3},$$

$$\mathbf{y_1(x) = x^3 - 0.1023508743\, x^{7/6} + L_x^{-1}[f_0(x)]}, \qquad (4.19)$$

$$\mathbf{y_1(x) = x^3 - 0.1023508743\, x^{7/6} + I^{2/3}(\textstyle\int_0^1 \sqrt{x}\, t[t^3]^2 dt},$$

$$\mathbf{y_1(x) = x^3},$$

$$\mathbf{y_{n+1}(x) = 0.1023508743\, x^{7/6} + L_x^{-1}[y_n(x)] = x^3,\ \ n \geq 1}, \qquad (4.20)$$

in similarly view equation (4.18) it is obtained , $y(x) = x^3$, where it is the exact solution of equation (4.10).

The solution according to (HPM)

$$\mathbf{D^{2/3}\, y(x)} = \frac{81\, x^{7/3}\sqrt{3}\,\Gamma(2/3)}{28\qquad \pi} - \frac{\sqrt{x}}{8} + \int_0^1 \sqrt{x}\, t\, [y(t)]^2\, dt, \qquad (4.15)$$

According to (3.16) we construct the following homotopy

$$\mathbf{D^{2/3}\, y(x) = p\left(g(x) + \textstyle\int_0^1 \sqrt{x}\ t[y(t)]^2 dt\right)}. \qquad (4.21)$$

Substituting (3.13) into (4.21),

$$\mathbf{p^0:\ \ D^{2/3} y_0(x) = 0},$$

$$\mathbf{p^1:\ \ D^{2/3}\, y_1(x) = g(x) + \textstyle\int_0^1 \sqrt{x}\ t\,[y_0(t)]^2 dt},$$

$$\mathbf{p^2:\ \ D^{2/3}\, y_2(x) = \textstyle\int_0^1 \sqrt{x}\ t\,[2y_0(t)\, y_1(t)]dt},$$

$$\mathbf{p^3:\ \ D^{2/3} y_3(x) = \textstyle\int_0^1 \sqrt{x}\ t\,\left[2y_0(t)y_2(t) + \left(y_1(t)\right)^2\right] dt},$$

$$\mathbf{p^4:\ \ D^{2/3} y_4(x) = \textstyle\int_0^1 \sqrt{x}\ t\,[2y_0(t)y_3(t) + 2y_2(t)y_1(t)]dt},$$

by applying the operators $I^{2/3}$to the above sets we obtain:

$$\mathbf{y_0(x) = 0},$$

$$\mathbf{y_1(x) = x^3 - 0.1023508743\, x^{7/6}},$$

$\mathbf{y_2(x) = 0}$,

$\mathbf{y_3(x) = 0.0775011686\, x^{7/6}}$,

$\mathbf{y_4(x) = 0}$,

Therefor the approximations to the solution of equation (4.14) will be determined as:

$\mathbf{y(x) = \sum_{i=0}^{\infty} y_i(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) + \cdots}$,

$\mathbf{y(x) = \cong x^3 - 0.02520075744\, x^{7/6}}$.

Table 3, Figure 5 and 6 shown the numerical result of example 4.3

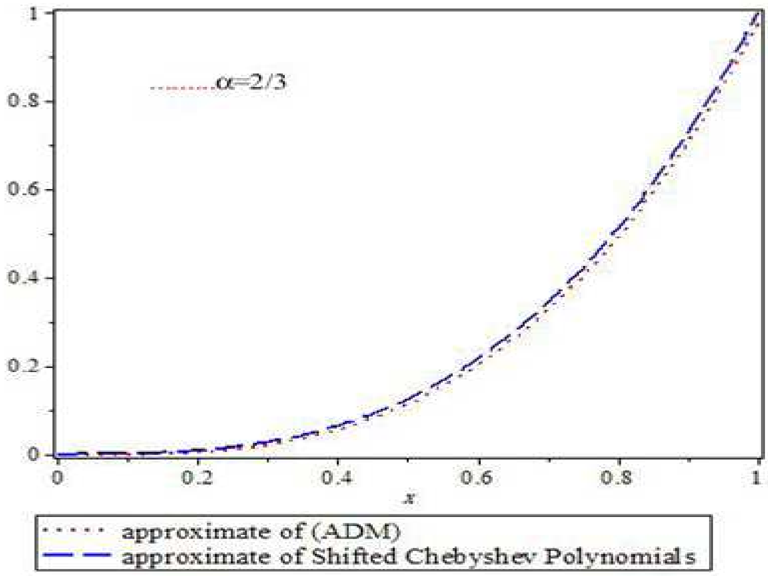

Figure 5: Numerical Result of Example 4.3

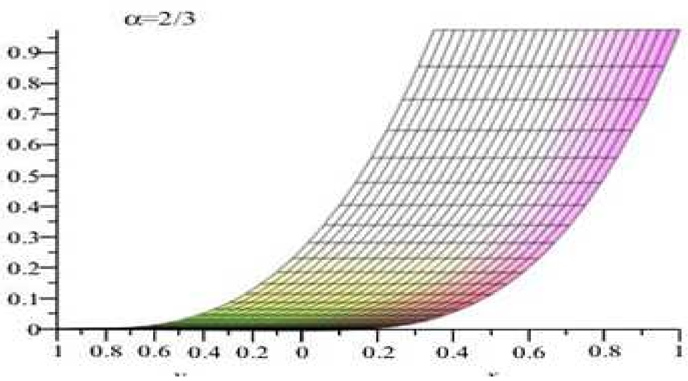

Figure 6: Approximate Solution of Example 4.3

Table 3 Indicate the amount of error in Example 4:3

x	exact = (MV IM)	Approximant by(HPM)	Error of(HPM)
0.1	0.001	- 0.000716907618	0.001716907618
0.2	0.008	0.004145672716	0.003854327284
0.3	0.027	0.02081430658	0.00618569342
0.4	0.064	0.05534732781	0.00865267219
0.5	0.125	0.1137743388	0.0112256612
0.6	0.216	0.2021135878	0.0138864122
0.7	0.343	0.3263775651	0.0166224349
0.8	0.512	0.4925754077	0.0194245923
0.9	0.729	0.7067141165	0.0222858835
1	1.000	0.9747992426	0.0252007574

CONCLUSIONS

From the above result we find that the modification of He' s variational iteration method (MVIM), is better than He s homotopy perturbation method (HPM), the results obtain by using Maple 16 .

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